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Note on Eigenvectors of a Renormalization Transformation

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It is generally assumed[1,2] that the eigenvalues $\lambda_\nu(b)$ of a linear renormalization transformation $\mathbf{R}_b$, defined by

$$\mathbf{R}_b \mathbf{e}_\nu = \lambda_\nu(b)\, \mathbf{e}_\nu \tag{1}$$

depend on the scaling factor b via $\lambda_\nu(b) = b^{y_\nu}$ (with y_ν independent of b), whereas the eigenvectors $\mathbf{e}_\nu$ do not. To our knowledge, there is no proof for the latter assumption in the literature. In this note we show that for nondegenerate eigenvectors of a renormalization transformation there is indeed no dependence on the scaling factor b. Of course, the arguments hold only for exact renormalization transformations, whereas eigenvectors of approximate transformations in general will depend on b.

The proof is made by contradiction: We assume for the moment that the eigenvectors do depend on b and demonstrate that this assumption leads to a contradiction for problems with nondegenerate eigenvectors.

We suppose that we have found exact renormalization transformations $\mathbf{R}_b$ and $\mathbf{R}_{b'}$ for two different scaling factors b and b', which in general are real numbers. Thereby we consider the same parameter space for both scaling factors, which does not represent any restriction, because we always can adjust the spaces by introducing additional couplings as required. We may approximate the ratio b/b' with arbitrary accuracy by m/n with $m, n \in Z$. This approximation is not critical because the renormalization transformation depends on the scaling factor b in a continuous manner. We then may write

$$\mathbf{R}_b^n = \mathbf{R}_{b'}^m \tag{2}$$

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The eigenvectors and eigenvalues of $\mathbf{R}_b$ and $\mathbf{R}_{b'}$ are $\mathbf{e}_\nu$ and $\mathbf{e}_{\nu'}$ as well as λ_ν and $\lambda_{\nu'}$, respectively. According to Eq. (1), application of $\mathbf{R}_b^n$ to an arbitrary eigenvector $\mathbf{e}_k$ yields $\mathbf{R}_b^n \mathbf{e}_k = \lambda_k^n \mathbf{e}_k$. We may represent $\lambda_k^n \mathbf{e}_k$ by the set of eigenvectors $\mathbf{e}_{\nu'}$, assuming for the moment that they are different from the eigenvectors $\mathbf{e}_\nu$:

$$\lambda_k^n \mathbf{e}_k = \sum_{\nu'} c_{\nu'} \mathbf{e}_{\nu'} \tag{3}$$

Applying on the left-hand side of this equation $\mathbf{R}_b^n$ and on the right-hand side $\mathbf{R}_{b'}^m = \mathbf{R}_b^n$ because of Eq. (2), we obtain

$$\lambda_k^n \lambda_k^n \mathbf{e}_k = \sum_{\nu'} c_{\nu'} \lambda_{\nu'}^m \mathbf{e}_{\nu'} \tag{4}$$

Inserting (3) into (4) gives

$$\sum_{\nu'} c_{\nu'} \lambda_k^n \mathbf{e}_{\nu'} = \sum_{\nu'} c_{\nu'} \lambda_{\nu'}^m \mathbf{e}_{\nu'} \tag{5}$$

resulting in $\lambda_k^n = \lambda_{\nu'}^m$ for all ν', for which $c_{\nu'} \neq 0$.

Obviously our assumption that the eigenvectors depend on b yields the result that all eigenvalues corresponding to nonvanishing $c_{\nu'}$ in the expansion (3) are degenerate. This is in contrast to our initial supposition of nondegenerate eigenvalues. The contradiction is resolved if there is only one nonvanishing coefficient $c_{\nu'}$. According to Eq. (3), this means (for normalized eigenvectors) that $\mathbf{e}_{k'} = \mathbf{e}_k$ for all k, i.e., the eigenvectors do not depend on the scaling factor b.

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